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ON THE THEORY OF THE RECTILINEAR OSCILLATOR.

By Edwin Bidwell Wilson.

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Introduction.

Planck's rectilinear oscillator is of great importance in theoretical physics. To grant this it is not necessary to assert a belief in the physical existence of the oscillator, much less to assert Planck's belief in such existence. The Carnot engine may not exist; but the concept has proved so fruitful as to be fundamental in the general field of thermodynamics. The (perfect) semipermeable membrane may be a figment of the imagination; but, in special fields in thermodynamics, reasoning based on the membrane is of crucial importance. And in that branch of thermodynamics which deals with radiant energy, the oscillator has held the position of cornerstone. Moreover, as in the case of the Carnot engine and semipermeable membrane, there does exist in physics a (more or less imperfect) prototype of the oscillator — the radiator or resonator of Hertz and his followers in radiotelegraphy.

In addition to the thermodynamic importance there is an interest attaching to the oscillator as the simplest model of the atom. Hypothetical models of the atom vary greatly; some are founded on statical configurations, others on dynamic equilibrium, some call for a number of corpuscles per atom comparable with the atomic weight, others for a number far greater than this, some represent the positive charge as widely spread out with the negative corpuscles imbedded in it, others concentrate the positive charge at a point. It is not necessary to take the oscillator very seriously as a model of an atom; all that is useful is to observe that by its means energy is absorbed and emitted and that thus it performs the functions of matter in establishing a temperature equilibrium in a closed field of radiant energy.

The object of this article is to make some comments on the equation which defines the motion of the oscillator. These comments are obvious at first sight to any one who examines the differential equations used in the theory of the oscillator, but as I cannot find them in the literature, I venture to print them at this late day.

1. **The differential equations.** Let us consider first some of the

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prime facts or hypotheses underlying the analysis. These suppositions are as follows:

1°. The oscillator executes in the main a simple harmonic motion. This leads to the statement of the (approximate) equation

$$\frac{d^2x}{dt^2} + k^2x = 0, \tag{1}$$

where x is the coordinate defining the state of the system.

The basic reason for this assumption is that it is desirable for the oscillator in its function of emitter of energy to be as simple as possible, that is, to emit monochromatic radiation of a definite frequency; for our customary analysis resolves heterogeneous radiation into monochromatic elements.

(If we wish to assume that the oscillator consists of a widely spread uniform positive charge at rest and of a concentrated negative charge vibrating to and fro through the center of the positive charge, we may obtain a model of the oscillator which satisfies the condition that the force on the moving particle shall be proportional to the displacement; for if a particle moves within a uniform sphere and if the elemental attraction follows the Newtonian law, the resultant force is proportional to the distance of the particle from the center of the sphere.)

2°. The motion of the oscillator is not affected by frictional resistance. This leads to (or corresponds with) the omission of a possible corrective term proportional to the velocity dx/dt in (1).

Perhaps the best reason for omitting frictional terms is that there seems to be no necessity for complicating the mathematical equations or physical hypotheses by admitting friction. In the Hertz radiator there is frictional dissipation of energy due to the resistance of the circuit; in the idealised oscillator the resistance would naturally be omitted.

3°. The motion is damped by the emission of energy. If the form of (1) appropriate to the principle of energy be used, that is, the integrated form, the undamped motion would be characterized by the equation

$$\left(\frac{dx}{dt}\right)^2 + k^2x^2 = C \qquad \text{or} \qquad \frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + k^2x^2\right] = 0,$$

and the damping would then be inserted by a change to the form

$$\left(\frac{dx}{dt}\right)^2 + k^2x^2 = C - 2\int_0^t F dt \quad \text{or} \quad \frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + k^2x^2\right] = -2F, \tag{2}$$

where F is the rate of radiation.

The modification of the equation is thus based directly on the principle of energy. The form in which F is thrown depends somewhat on the point of view.

If the oscillator is considered as a Hertz dipol and the ordinary solution for the electromagnetic field set up by this radiator at large distances from its center is obtained, we find by the application of Poynting's theorem that a certain amount of energy is radiated off per period and, in particular, that the amount is proportional to the square of the acceleration. Then the equation, written for a complete integrated period T is

$$\int_{t_0}^{t_0+T} \left\{ \frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + k^2x^2 \right] + 2\mu \left(\frac{d^2x}{dt^2}\right)^2 \right\} dt = 0.$$

An integration by parts transforms the third term. Whence

$$\int_{t_0}^{t_0+T} \left\{ \frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + k^2x^2 \right] - 2\mu \frac{dx}{dt}\frac{d^3x}{dt^3} \right\} dt + 2\mu \left[\frac{dx}{dt}\frac{d^2x}{dt^2} \right]_{t_0}^{t_0+T} = 0. \tag{3}$$

Now if t_0 be an epoch for which either the velocity or the acceleration vanishes, the last term drops out. Under these conditions

$$\int_{t_0}^{t_0+T} \left\{ \frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + k^2x^2 \right] - 2\mu \frac{dx}{dt}\frac{d^3x}{dt^3} \right\} dt = 0.$$

The conclusion is then drawn that

$$\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + k^2x^2 \right] - 2\mu \frac{dx}{dt}\frac{d^3x}{dt^3} = 0. \tag{4}$$

Exceptions can be taken to this conclusion whether the argument is as above or as given in slightly different form by Planck.[2] It by no means follows that, because the integral of a function over a complete period from t_0+nT to $t_0+(n+1)T$ vanishes, the integrand must therefore be identically null.

What we may infer is that

$$\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + k^2x^2 \right] - 2\mu \frac{dx}{dt}\frac{d^3x}{dt^3} = a(t)$$

where $a(t_0+(n+1)T) - a(t_0+nT) = 0$. There is nothing at all to

2 Planck, Vorlesungen über die Theorie der Wärmestrahlung, first edition, 110 (1906). Abraham, Theorie der Elektrizität, second edition, 70 (1908). Rüdenberg, Annalen der Physik, **25**, 346–466 (1908). (In his second edition, Planck introduces the quantum hypothesis early and omits this derivation of the equation of the oscillator.)

suggest that the function $a(t)$ has a negligible effect upon the motion. It therefore follows that (4) or the equation

$$\frac{d^2x}{dt^2} + k^2x - \mu \frac{d^3x}{dt^3} = 0, \tag{5}$$

obtained therefrom by discarding the factor dx/dt, is at best only approximate. Concerning the degree of approximation we may perhaps be willing to admit that the error is always extremely small; but we should be very chary about admitting that the error is necessarily small relative to the terms retained.

That (5) is only an approximate result is clearly brought out by Planck in his derivation, but the clearness is somewhat obscured by a footnote referring to Abraham's derivation in a manner which implies, though it does not state, that the latter proof is exact. Abraham gives in the first instance essentially the proof which we have given above. He goes on, however, to show by a discussion of the physical dimensions of the terms that the equation (5) is the only possible one. What he really proves is that under certain hypotheses (5) is the only possible linear equation of the form

$$\frac{d^2x}{dt^2} + k^2x = \sum_i c_i \frac{d^iv}{dt^i}.$$

The physical dimensions are equally well satisfied by the non-linear equation

$$\frac{dx}{dt}\left[\frac{d^2x}{dt^2} + k^2x\right] + \mu\left(\frac{d^2x}{dt^2}\right)^2 = 0, \tag{6}$$

which follows naturally and immediately from (2), that is, from the principle of energy, provided it is assumed that the rate of radiation is proportional *at each instant* to the square of the acceleration, instead of merely proportional thereto when integrated over a complete period, and which would arise from the equation in integrated form, provided one were willing to apply to it directly the reasoning which is applied after an integration by parts.

Lorentz has also derived (5) by a method which throws emphasis on the approximate nature of the result.[3] That he regards the equation with some suspicion may be inferred from his statement of a later date[4]: Eine Gleichung "die übrigens den Mathematikern noch manche Frage darböte."

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3 See, for example, his Theory of Electrons, 251, (1909).

4 Physikalische Zeitschrift, **11**, 1250 (1910) It was this remark which so strengthened my own suspicions as to lead me to take up the question some time ago.

The calculation of the instantaneous rate of radiation of energy from a moving electron, as contrasted with the integrated rate hitherto used, has been carried out by a number of authors in different ways. The rate is almost universally found proportional to the square of the acceleration when the square of the velocity is negligible relative to the square of the velocity of light.[5] The mathematical statement of this result is contained in (6), and this equation therefore might more properly be taken as representative of the motion of the oscillator than the equation (5),— provided, of course, that we are willing to use the electron model of the oscillator.

A comparison of (5) and (6) will throw some light on the legitimacy of the approximations used in deriving (5). The terms to be compared are

$$-\mu \frac{dx}{dt}\frac{d^3x}{dt^2} \quad \text{and} \quad \mu\left(\frac{d^2x}{dt^2}\right)^2.$$

On the supposition that the oscillator executes in the main a simple harmonic motion, the signs of the two terms are the same. But the phases of the magnitudes are very different; for the first term is a maximum as the particle passes through the position of equilibrium and vanishes in the positions of extreme elongation, whereas the second term is a maximum in the extreme positions and vanishes in the position of equilibrium. The average magnitudes of the terms are the same, but it is difficult to see how two terms of equal average magnitude could be more different than these two in their effect upon the equation of motion.

We shall now discuss the two forms (5) and (6), the first rather briefly, because it has been tolerably well treated, the second more in detail, because it seems to have been neglected.

2. **Discussion of (5).** As we have to deal with cubic equations and their (approximate) solutions, it is necessary to determine the order of magnitude of the coefficients and it is desirable to put the equations into their simplest form. If m is the mass of the vibrating particle, e the charge in electrostatic units, c the velocity of light, and if we adopt the customary coefficient for the radiation term, we have

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5 For two different lines of deduction see J. J. Thomson, Electricity and Matter, chap. 3 (1904), and Wilson and Lewis, The Space-time Manifold of Relativity, These Proceedings, **48**, 389–507 (1912), with especial reference to p. 480. A. Macdonald in his Electric Waves, 75 (1902), finds a steady rate of radiation from the oscillator.

$$m\left(\frac{d^2x}{dt^2}+k^2x\right)-\frac{2}{3}\frac{e^2}{c^3}\frac{d^3x}{dt^3}=0$$

or

$$m\frac{dx}{dt}\left[\frac{d^2x}{dt^2}+k^2x\right]+\frac{2}{3}\frac{e^2}{c^3}\left(\frac{d^2x}{dt^2}\right)^2=0.$$

As numerical values we may assume

$$e=4.7\ 10^{-10},\qquad m=9\ 10^{-28},\qquad c=3\ 10^{10}.$$

Hence

$$\frac{2}{3}\frac{e^2}{mc^3}=6\ 10^{-24}.$$

Substitute $t=a\tau$ and choose $a^2k^2=1$. Then

$$\frac{d^2x}{d\tau^2}+x-6\,k\,10^{-24}\frac{d^3x}{d\tau^3}=0$$

and

$$\frac{dx}{d\tau}\left[\frac{d^2x}{d\tau^2}+x\right]+6\,k\,10^{-24}\left(\frac{d^2x}{d\tau^2}\right)^2=0.$$

Now $k=2\pi c/\lambda$, where λ is the wave length. The observed range of λ in the spectrum is from 10^{-5} to $3\ 10^{-2}$ with λ about $5.5\ 10^{-5}$ in the center of the visible spectrum. The range for the coefficient of the last term in the equations is therefore from about 10^{-7} in the ultra-violet to about $4\ 10^{-11}$ in the infra-red with $2\ 10^{-8}$ for the value near the middle of the range of visibility.

The numerical forms of the equations are therefore

$$\frac{d^2x}{d\tau^2}+x-\kappa\frac{d^3x}{d\tau^3}=0 \tag{5'}$$

and

$$\frac{dx}{d\tau}\left[\frac{d^2x}{d\tau^2}+x\right]+\kappa\left(\frac{d^2x}{d\tau^2}\right)^2=0 \tag{6'}$$

with

$$10^{-7}>\kappa>4\ 10^{-11}\qquad\text{or}\qquad\kappa=2\ 10^{-8},$$

the latter being taken as a sort of standard value.

(It is not necessary to make the special assumptions involved in the electron model of the oscillator if we desire merely to determine the order of magnitude of κ. If (5′) be replaced by

$$\frac{d^2x}{d\tau^2}+\kappa\frac{dx}{d\tau}+x=0, \tag{7}$$

which is nearly equivalent to it, since x and d^2x/dt^2 are nearly equal, the amplitude of the motion is seen to suffer a reduction in the ratio

$$e^{-\frac{1}{2}\kappa 2\pi n} : 1$$

during n complete oscillations. We may now use experiments on the distinctness of interference fringes for different path-lengths to obtain an estimate for κ, and we find that $2\ 10^{-8}$ is an entirely reasonable value. See Abraham, loc. cit.)

The complete solution of (5′) will be of the form

$$x = C_1 e^{\alpha\tau} + e^{-\beta\tau}\,(C_2 \cos\gamma\tau + C_3 \sin\gamma\tau),$$

where $\alpha, -\beta \pm \gamma i$ are the roots of the cubic

$$r^2 + r - \kappa r^3 = 0.$$

The root α is positive and of great magnitude, approximately $1/\kappa$, say, $\frac{1}{2}\ 10^8$. As Planck says, this root has no physical significance.[6] This means that for every determination of the constants C_1, C_2, C_3, the value of C_1 must vanish and the solution of the equation reduce to

$$x = e^{-\beta\tau}\,(C_2 \cos\gamma\tau + C_3 \sin\gamma\tau).$$

It is certainly unfortunate that we are forced to discard so arbitrarily one root of an equation which we have labored so carefully to establish. Moreover, as the solution of the equation now depends only on two constants, the elimination of these constants by differentiation will give an equation of the second order which accounts satisfactorily for the phenomenon as far as its mathematical side is concerned. This equation, which is almost identical with (7), contains a friction term, it is true, and is unsatisfactory physically because of this fact. But there is no real reason for considering it as less satisfactory than (5′) which has a physically meaningless root and which moreover, while ostensibly allowing for radiation on a physical basis, actually requires the rate of radiation to be maximum where there is a general agreement that it should be minimum, and minimum where it should be maximum.

3. **Discussion of (6).** We turn next to the equation (6′) and, for the sake of brevity, write it as

$$\kappa f^2 + vf + vx = 0, \tag{8}$$

f and v designating respectively acceleration and velocity when the

[6] If we make clear the assumption of the quasi-stationary state as a preliminary condition in our approximations which lead to the equation, the large root is sure to be inadmissible by virtue of our assumption.

unit of time is the fraction $\lambda/(2\pi c)$ of the second. This equation differs very minutely from (5′) or (7), but there is no reason to believe that the motion defined by the equation should resemble very closely the motion defined by those. As the integration of the non-linear equation of the second order is not simple, we shall first examine the equation to find certain elementary qualities of the solution.

A. When the velocity vanishes, the acceleration also vanishes.

B. When the acceleration vanishes, either the velocity or the displacement vanishes.

C. In the neutral position $x = 0$, either the acceleration vanishes or the acceleration and velocity satisfy the relation $\kappa f + v = 0$.

D. Arbitrarily assumed (initial) values of the acceleration and position determine the velocity, and similarly values of the acceleration and velocity determine the position, but assumed values of the velocity and position allow the acceleration either of two values

$$f = -\frac{v}{2\kappa} \pm \frac{1}{2\kappa}\sqrt{v^2 - 4\kappa v x}.$$

E. Values of v and x which make $v^2 - 4\kappa v x$ negative are not admissible. Hence if x is positive, v must either exceed $4\kappa x$ or be negative; and if x is negative, v must be positive or less than $4\kappa x$ (algebraically).

F. The case of x positive and v greater than $4\kappa x$ leads to a *point d'arrêt*. For under these conditions f is negative, no matter which sign is taken before the radical, and the velocity must be decreasing as x increases; when v becomes equal to $4\kappa x$, the acceleration becomes imaginary, and so does the motion. The same arises under the case of x negative and v less than $4\kappa x$.

G. The case x positive and v negative also leads to a point d'arrêt. For under these conditions if the negative sign is taken with the radical, then f is negative and the magnitude of v is increasing as x diminishes toward zero, the particle will reach the origin with a definite velocity and pass through to the negative values of x where it falls under the case *F;* and if the positive sign is taken with the radical, so that f is positive, the particle is slowing down, and we have to consider two possibilities, first where the particle reaches the origin before its velocity vanishes and where the argument just given will then again be applicable, second where the particle comes to rest before reaching the origin only to start on again with a negative f and fall under the first supposition of this case *G*. A similar discussion may be given for the hypothesis x negative and v positive. (There

is a special case in which the particle just comes to rest at the origin, remaining there in a sort of unstable equilibrium.)

The consequences of this discussion of the second type of equation — the equation (6) founded on the radiation formula generally adopted — may be formulated in the unsatisfactory statement that, except for two special possibilities, the motion runs into a point d'arrêt where at a certain time and in a certain position, with a definite velocity and acceleration neither of which vanishes, the motion suddenly becomes imaginary. This is very disconcerting; it is at direct variance with our accepted notions of what happens in mechanical or electrodynamic systems.

It is also a consequence that the oscillator governed by equation (6) cannot oscillate.

Unfamiliar as this result may sound to those who have a feeling that a slight change in the differential equation introduces only a slight change in the motion, we must nevertheless admit the result and develop the contrary feeling that a slight change may entirely alter the type of motion. This fact has been brought out by Borel in a paper just printed [7] wherein it is shown that the equation $v^2 + x^2 = a^2 + \lambda t$, no matter how small the constant λ, does not define an oscillatory motion, nor does the motion approach the oscillatory type as λ approaches zero. Any one might admit that, in the course of a long time, λt would become important and the type of motion be changed; but this is not the true explanation, for, no matter how small the constant λ, the motion departs from the simple harmonic type from the time $t = \pi/2$, that is, after executing a quarter oscillation.

4. **The integrals of (6).** We shall now turn to the integration of the equation

$$\frac{dx}{d\tau}\left(\frac{d^2x}{d\tau^2} + x\right) + \kappa\left(\frac{d^2x}{d\tau^2}\right)^2 = 0. \qquad (6')$$

The first thing we shall treat is motion from a position of rest. In

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7 Borel, Annali di Matematica (**3**), **21**, 225, (1913). The equation of Borel corresponds to a constant radiation of energy. The solutions of the equation do not exhibit the point d'arrêt, but are continuous. It has seemed to me, notwithstanding the publication of Borel's paper, and perhaps even on account of it, that the printing of my own results would be of interest. The comments on Borel's paper, which he prints as a footnote to a brief account of his results in his new book, Introduction Géométrique à Quelques Théories Physiques, 107 (1914) appear less applicable to the results of our present work.

this case v and f are both zero. Let us therefore assume a solution for x as a series

$$x = a(1 + \alpha\tau^3 + \beta\tau^4 + \gamma\tau^5 + \ldots)$$

in τ. The equations for the determination of the coefficients are

$$36a^2\kappa + 3a = 0, \qquad 144a\beta\kappa + 18a^2 + 4\beta = 0,$$
$$(144\beta^2 + 240a\gamma)\kappa + 60a\beta + 5\gamma = 0, \ldots.$$

Hence

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = 0, \ldots.$$

or

$$\alpha = -\frac{1}{12\kappa}, \qquad \beta = \frac{1}{64\kappa^2}, \qquad \gamma = -\frac{11}{3840\kappa^3}, \ldots.$$

Thus one solution for x is $x = a$, the particle remains at rest in the initial position, and the other solution is

$$x = a\left(1 - \frac{\tau^3}{12\kappa} + \frac{\tau^4}{64\kappa^2} - \frac{11\tau^5}{3840\kappa^3} + \ldots.\right), \tag{9}$$

the particle moves with an acceleration toward the origin, as was foreseen under G above.

If this solution in series is valid, that is, if the series converges, it is at any rate of no value for purposes of calculation unless τ is exceedingly small, that is, for intervals of time which are very small compared with the period of monochromatic light ($\tau = 2\pi$). The value of the series is first to show how the motion starts and second to show that there is a possibility of motion, that is, that there is a solution of the equation other than the solution $x = a$ with the given initial conditions. In discussing this question we shall follow the method given by Picard.[8]

Let the equation (6′) of the second order be written as two simultaneous equations of the first order, namely,

$$\kappa\left(\frac{dv}{d\tau}\right)^2 + v\left(\frac{dv}{d\tau} + x\right) = 0, \qquad \frac{dx}{d\tau} - v = 0. \tag{10}$$

These may be expressed in solved form as

$$\frac{dx}{d\tau} = v, \qquad \frac{dv}{d\tau} = -\frac{v}{2\kappa} \pm \frac{1}{2\kappa}\sqrt{v^2 - 4\kappa vx}.$$

The fundamental theorem on the existence of integrals fails for those

[8] E. Picard, Traité d'Analyse, **3**, chap. 3.

values of the variables for which two values of the derivatives become equal. In this case the failure is when

$$v^2 - 4\kappa v x = 0.$$

This equation factors, and there are therefore two surfaces

$$v = 0, \qquad v = 4\kappa x$$

in the space (v, x, τ) which are singular in the sense that the fundamental existence theorem fails.

Of these surfaces one, namely, $v = 0$, satisfies the differential equation (6′) and is therefore singular also in the sense that it is the envelope of particular solutions. It is possible to put the particular solutions in evidence. For let x be positive so that $v \leqq 0$. Introduce a new variable $v = -z^2$ and let the sign of z be so chosen that z increases with τ. We have then the equation

$$-2z\frac{dz}{d\tau} = -\frac{z^2}{2\kappa} \pm \sqrt{z^4 + 4\kappa z^2 x}.$$

In cancelling the variable z the double sign goes out. For since f is decreasing with τ (see G above) and since

$$f = \frac{dv}{d\tau} = -2z\frac{dz}{d\tau},$$

it follows that dz/dt is positive. We therefore have to integrate the equations

$$\frac{dz}{d\tau} = \frac{z}{4\kappa} + \frac{1}{4\kappa}\sqrt{4\kappa x + z^2}, \qquad \frac{dx}{d\tau} = -z^2,$$

with the initial conditions $z = 0$, $x = a < 0$, to obtain the particular solutions which are tangent to the singular solution $v = 0$. For these equations the existence theorems are applicable. As a solution exists, it must be (9) which we found by undetermined coefficients.

The other singular solution $v = 4\kappa x$ is not singular in the sense that it is enveloped by particular solutions but in the sense that it is a cusp locus for particular solutions (see Picard, loc. cit.). Without following Picard through the individual steps of the work we may write down the result. If v_0 and x_0 are two values satisfying the relation $v = 4\kappa x$ at $\tau = \tau_0$, the solution may be expanded into the form

$$x = x_0 + \alpha(\tau_0 - \tau) + \beta(\tau_0 - \tau)^{\frac{3}{2}} + \gamma(\tau_0 - \tau)^2 + \delta(\tau_0 - \tau)^{\frac{5}{2}} + \ldots$$
$$v = v_0 + \alpha'(\tau_0 - \tau) + \beta'(\tau_0 - \tau)^{\frac{3}{2}} + \gamma'(\tau_0 - \tau)^2 + \delta'(\tau_0 - \tau)^{\frac{5}{2}} + \ldots$$

Now let us take the case $x_0 > 0$, $f_0 = -v_0/2\kappa$. The constant β is zero, and

$$x = x_0 - v_0(\tau_0 - \tau) - \frac{v_0}{4\kappa}(\tau_0 - \tau)^2 + \delta(\tau_0 - \tau)^{\frac{5}{2}} + \ldots\ldots,$$

$$v = v_0 + \frac{v_0}{2\kappa}(\tau_0 - \tau) + \frac{5}{2}\delta(\tau_0 - \tau)^{\frac{3}{2}} + \ldots$$

The value of δ, which does not vanish, may readily be found, if desired.

This solution brings out the fact very clearly that the motion runs into a point d'arrêt. The curve, to be sure, has a cusp, but the expansion is in powers of $\sqrt{\tau_0 - \tau}$, which means that the time cannot exceed τ_0. If we could imagine that time satisfies the same possibilities of reversal of direction as space, we could interpret this cuspidal motion, but as time can flow only in one direction, the motion must be interpreted as having a point d'arrêt. This does not mean that the two parts of the geometric locus cannot be interpreted, but that they must be representative of two different motions terminating in the same point d'arrêt.

We may indeed trace the motion back from the point d'arrêt along the two branches. We have

$$f = -\frac{v}{2\kappa} \pm \frac{1}{2\kappa}\sqrt{v^2 - 4\kappa v x}.$$

The acceleration is negative for both signs before the radical; x has increased up to x_0 at $\tau = \tau_0$ and v has decreased to v_0. In following the motion backward through time we find that x decreases and v

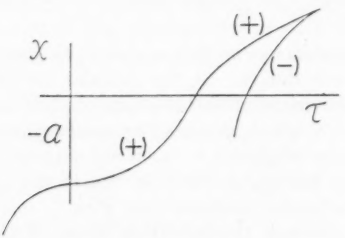

FIGURE 1.

increases. For the motion with the positive sign before the radical the retardation is less than for the motion with the negative sign. Hence in going backward, the former motion gives larger values of x

and smaller values of v than the latter motion for the same value of $\tau_0 - \tau$. When the $(+)$ motion passed the origin, the acceleration was zero; when the $(-)$ motion passed the origin, the acceleration was negative and twice as large as at the cusp. The $(+)$ motion may be traced back to a point where the particle comes to rest for a certain negative value of x; this corresponds to the solution (9), and the motion may be traced still further back, the velocity and retardation increasing all the while. The $(-)$ motion may likewise be traced back through the origin $x = 0$, the velocity and retardation increasing.

The curve which shows roughly the relation between x and τ is given in Figure 1. If we call the curve which corresponds to the positive sign $x = a\Phi_{(+)}(\tau)$ and the curve which corresponds to the negative sign $x = a\Phi_{(-)}(\tau)$, we may at once write the complete solution of the differential equation (6′), namely,

$$x = C_1\Phi_{(+)}(\tau - C_2), \qquad x = C_1\Phi_{(-)}(\tau - C_2),$$

with the singular solutions $x = C$. The first solution could have been written at once from (9); the particle is at rest at $x = C_1$ when $\tau = C_2$.

The way in which the constants of integration enter into the solution (which could be foreseen from the differential equation itself inasmuch as the equation admits the two-parameter group $x' = x + a$, $\tau' = \tau + a$) shows that the motion, though not periodic, has definite intervals of time independent of the constants of integration associated with it — the time from rest to the origin and the times (two possibilities) from the origin to the point d'arrêt.[9]

5. Numerical Calculations. To ascertain these periods of time we must integrate the equation (6′). Let

$$a = \alpha e^{-\int u d\tau}, \qquad \frac{dx}{d\tau} = -xu, \qquad \frac{d^2x}{d\tau^2} = x\left(u^2 - \frac{du}{d\tau}\right). \tag{11}$$

The equation in u is

$$\kappa\left(u^2 - \frac{du}{d\tau}\right)^2 - u\left(u^2 - \frac{du}{d\tau}\right) - u = 0$$

or

$$\frac{du}{d\tau} = u^2 - \frac{u}{2\kappa} \pm \frac{1}{2\kappa}\sqrt{u^2 + 4\kappa u}.$$

Suppose $x = a < 0$, $v = 0$, $f = 0$ are the initial conditions. Then $u = 0$. Moreover, as f becomes positive, $du/d\tau - u^2$ becomes

[9] These latter would not appear in the case Borel treated.

positive. Hence the positive sign must be taken with the radical. When $x = 0$, u has become positively infinite. The time of fall to the origin is therefore

$$\int_0^T d\tau = T = \int_0^\infty \frac{du}{u^2 - \frac{u}{2\kappa} + \frac{1}{2\kappa}\sqrt{u^2 + 4\kappa u}}. \tag{12}$$

Introduce the substitution

$$y^2 = \frac{u}{u + 4\kappa}, \qquad u = \frac{4\kappa y^2}{1 - y^2}. \tag{13}$$

Then

$$T = \int_0^1 \frac{4\kappa\, dy}{8\kappa^2 y^3 + (1 - y^2)(1 - y)}.$$

The integration may be performed by partial fractions. It is necessary to factor the denominator approximately. One root is nearly equal to -1, the other two are imaginary. The real root may be developed in powers of κ^2. The first approximation gives

$$y = -1 + 2\kappa^2.$$

The denominator is factorable as

$$(1 + 8\kappa^2)\ (y + 1 - 2\kappa^2)\ [y^2 - 2(1 - 5\kappa^2)y + 1 - 6\kappa^2].$$

The reduction of the integrand to partial fractions gives

$$\kappa\,(1 - 2\kappa^2)\left[\frac{1}{y + 1 - 2\kappa^2} - \frac{y - (1 - 5\kappa^2) - (2 - 7\kappa^2)}{y^2 - 2\,(1 - 5\kappa^2)\,y + 1 - 6\kappa^2}\right].$$

For the integration

$$\tau = \kappa\,(1 - 2\kappa^2)\left[\tfrac{1}{2}\log\frac{(y + 1 - 2\kappa^2)^2}{y^2 - 2\,(1 - 5\kappa^2)\,y + 1 - 6\kappa^2} + \frac{2 - 7\kappa^2}{2\kappa}\tan^{-1}\frac{y - 1 + 5\kappa^2}{2\kappa}\right].$$

Substituting the limits 0 and 1, we have

$$\tau = \kappa\,(1 - 2\kappa^2)\left[\log\left(\frac{1 - \kappa^2}{\kappa}\,\frac{1 - 3\kappa^2}{1 - 2\kappa^2}\right) + \frac{2 - 7\kappa^2}{2\kappa}\left(\tan^{-1}\frac{5\kappa}{2} + \tan^{-1}\frac{1 - 5\kappa^2}{2\kappa}\right)\right].$$

The κ^2 is now everywhere negligible, and

$$T = \frac{\pi}{2} + \kappa\log\frac{1}{\kappa} + \frac{\kappa}{2}.$$

As κ is in the neighborhood of $2\ 10^{-8}$, the difference between τ and $\pi/2$ is about $4\ 10^{-7}$.

We desire next to follow the motion through the origin and on to the point d'arrêt.[10] Construct the Riemann surface for the integrand in (12). The surface has branch points at $u = 0$ and $u = -4\kappa$, and a junction line between them. We have let u trace the real axis from 0 to ∞; now let u turn through a large circuit in the upper half-plane

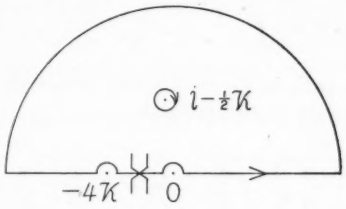

FIGURE 2.

and come down to the other end of the axis of reals. No time elapses during this process, for approximately $d\tau = du/u^2$. The variable x merely swings around the origin on an infinitesimal semi-circumference and is ready to start off on positive values as u comes in along the negative real axis. That the branch point $u = -4\kappa$ corresponds to the point d'arrêt for the motion is readily surmised, and may be proved from the defining equations (11).

The time from the origin to the point d'arrêt is therefore

$$T' = \int_{\infty}^{-4\kappa} \frac{du}{u^2 - \frac{u}{2\kappa} - \frac{1}{2\kappa}\sqrt{u^2 + 4\kappa u}}.$$

But with the substitution (13) the integrand becomes again the same:

$$T' = \int_1^{\infty} \frac{4\kappa\ dy}{8\kappa^2 y^3 + (1 - y^2)(1 - y)}.$$

Hence

$$T' = \kappa(1 - 2\kappa^2)\left[-\log\frac{1 - \kappa^2}{\kappa} + \frac{2 - 7\kappa^2}{2\kappa}\left(\frac{\pi}{2} - \tan^{-1}\frac{5\kappa}{2}\right)\right].$$

10 The simplest method seems to be one dependent on the theory of functions of a complex variable; for it enables us easily to pass around the singularity at the origin.

Again κ^2 is negligible and

$$T' = \frac{\pi}{2} - \kappa \log \frac{1}{\kappa} - \frac{5}{2}\kappa.$$

If we had desired merely the total time $T + T' = \pi - 2\kappa$ for the swing from rest to point d'arrêt, we could have obtained the result more directly by the theory of residues. The integral

$$T + T' = \int_0^{-4\kappa} \frac{du}{u^2 - \frac{u}{2\kappa} + \frac{u}{2\kappa}\sqrt{1 + \frac{4\kappa}{u}}},$$

taken around the path in Figure 2 which passes through infinity, is equivalent to the sum of two integrals

$$-\int_{-4\kappa}^{0} \frac{du}{u^2 - \frac{u}{2\kappa} + \frac{u}{2\kappa}\sqrt{1 + \frac{4\kappa}{u}}} + \int_{\circ} \frac{du}{u^2 - \frac{u}{2\kappa} + \frac{u}{2\kappa}\sqrt{1 + \frac{4\kappa}{u}}}, \quad (14)$$

where the first is taken along the direct path on the upper side of the junction line and where the second is taken around that pole of the integrand which lies in the upper half of the upper sheet of the Riemann surface. As the calculation of the position of this pole is necessary for the work of the next section we shall now check the above value of $T + T'$.

The denominator of the integrand vanishes when

$$u^4 - \frac{u^3}{\kappa} + \frac{u^2}{4\kappa^2} - \frac{u^2}{4\kappa^2}\left(1 + \frac{4\kappa}{u}\right) = 0.$$

This gives one zero at the branch point $u = 0$; the integral to $u = 0$, however, is finite as the root is simple and the branch point has two sheets. The other roots are determined by the cubic

$$-\kappa u^3 + u^2 + 1 = 0 = -\kappa(u - \tfrac{1}{\kappa} - \kappa)(u^2 + \kappa u + 1).$$

To terms in κ the approximation gives the three roots

$$u = \frac{1}{\kappa} + \kappa, \qquad u = \pm i - \frac{\kappa}{2}.$$

The first lies in the under sheet and does not concern us. The root $u = i - \frac{1}{2}\kappa$ lies in the upper half of the upper sheet, and the residue of the corresponding pole has to be found. The expansion

$$u^2 - \frac{u}{2\kappa} + \frac{u}{2\kappa}\left(1 + \frac{2\kappa}{u} - \frac{2\kappa^2}{u^2}\right)$$

of the denominator is legimate in the neighborhood of $u = i - \frac{1}{2}\kappa$. The denominator therefore becomes

$$u^2 + 1 - \frac{\kappa}{u} = \left(u - i + \frac{1}{2}\kappa\right)\left(u + i - \frac{1}{2}\frac{\kappa}{u} - \frac{\kappa i}{u}\right)$$

The residue is

$$\left.\frac{1}{u + i - \frac{1}{2}\kappa - \frac{\kappa i}{u}}\right]_{u = i - \frac{1}{2}\kappa} = \frac{1}{2i\,(1 + i\kappa)} = \frac{1}{2i}\,(1 - i\kappa).$$

The second integral in (14) is therefore $\pi\,(1 - i\kappa)$, which is $2\pi i$ times the residue.

To calculate the first integral in (14) we may rationalize:

$$-\int_{-4\kappa}^{0} \frac{u^2 - \frac{u}{2\kappa} - \frac{u}{2\kappa}\sqrt{1 + \frac{4\kappa}{u}}}{u^4 - \frac{u^3}{\kappa} - \frac{u}{\kappa}}\,du.$$

The first two terms of the denominator and the first of the numerator are of the order κ^2, or higher between -4κ and 0 and may be neglected. The result is

$$-\int_{-4\kappa}^{0} \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\kappa}{u}}\,du =$$

$$-\frac{1}{2}\left[u + \sqrt{u\,(u + 4\kappa)} + 4\kappa \log\left(\sqrt{u + 4\kappa} + 5u\right)\right]_{-4\kappa}^{0} =$$

$$-2\kappa - 2\kappa \log\,(\doteq i),$$

where the indeterminate sign and the indeterminate value of the logarithm must be properly chosen. This need not trouble us as the term is imaginary, and its only function is to cancel the imaginary part of $\pi(1 - i\kappa)$. The total value of $T + T'$ is thus seen to be $\pi - 2\kappa$ as previously determined.

We have next to find the value of $x = X$ at the point d'arrêt. This may be written as

$$X = a\,e^{-\int_{0}^{-4\kappa} \frac{u\,du}{u^2 - \frac{u}{2\kappa} + \frac{u}{2\kappa}\sqrt{1 + \frac{4\kappa}{u}}}},$$

where the integral is again taken over the roundabout path of Figure 2. We may write

$$-\int_0^{-4\kappa} \frac{u\,du}{u^2 - \frac{u}{2\kappa} + \frac{u}{2\kappa}\sqrt{1+\frac{4\kappa}{u}}} = \int_{-4\kappa}^{0} \frac{u\,du}{u^2 - \frac{u}{2\kappa} + \frac{u}{2\kappa}\sqrt{1+\frac{4\kappa}{u}}} - \int_{\circ} \frac{u\,du}{u^2 - \frac{u}{2\kappa} + \frac{u}{2\kappa}\sqrt{1+\frac{4\kappa}{u}}},$$

where the first integral on the right is taken along the junction line, and the second about $u = i - \frac{1}{2}\kappa$. The former is approximately

$$\int_{-4\kappa}^{0} u\left(\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{4\kappa}{u}}\right) du = \frac{1}{2}\int_{-4\kappa}^{0} u + \sqrt{u^2+4\kappa\,u}\,du,$$

which is of order κ^2 and vanishes to the present degree of approximation. The latter is

$$-2\pi i\left[\frac{u}{u+i-\frac{1}{2}\kappa-\frac{\kappa i}{u}}\right]_{u=i-\frac{1}{2}\kappa} = -2\pi i\,\frac{i-\frac{1}{2}\kappa}{2i\,(1+i\kappa)}$$

$$= -\pi i - \frac{1}{2}\pi\kappa.$$

The value of X is therefore,

$$X = -a\,e^{-\frac{1}{2}\pi\kappa} = -a\,(1-\tfrac{1}{2}\pi\kappa).$$

The point d'arrêt therefore lies slightly nearer the origin than the point of rest.

To obtain the amount of energy lost by radiation during the swing from the starting point of no velocity to the point d'arrêt we might evaluate the integral of $\kappa(d^2x/d\tau^2)^2$ as we have evaluated the integrals for the time and displacement. But it is simpler to figure the final kinetic energy and potential energy and to subtract their sum from the initial potential energy $\frac{1}{2}a^2$. The final velocity is $v = 4\kappa x$, with $x = X = -a(1-\frac{1}{2}\pi\kappa)$. The kinetic energy $\frac{1}{2}v^2$ is therefore negligible to the order of our approximations. The potential energy is $\frac{1}{2}x^2 = \frac{1}{2}(a^2 - \pi\kappa)$ which is less than $\frac{1}{2}a^2$ by the amount $\frac{1}{2}\pi\kappa$. Thus the energy radiated during the swing is essentially the same as that radiated during one swing on the assumption that the motion is practically simple harmonic.

We have now concluded a tolerably complete qualitative and quantitative discussion of the equation

$$m\frac{dx}{dt}\left(\frac{d^2x}{dt^2}+\kappa^2x\right)=-\frac{2}{3}\frac{e^2}{c^3}\left(\frac{d^2x}{dt^2}\right)^2, \tag{15}$$

which arises by the application of the principle of energy with the usual law of instantaneous radiation, and which would arise from the integrated form of the energy equation by the same reasoning as is applied, after a transformation by integration by parts, namely, that if an integral vanishes, its integrand also vanishes, to obtain the equation

$$m\left(\frac{d^2x}{dt^2}+\kappa^2x\right)=\frac{2}{3}\frac{e^2}{c^3}\frac{d^3x}{dt^3}. \tag{16}$$

It is probable that if the non-linear equation (15) had been easily integrable and had given a motion essentially simple harmonic and damped, the transformation to (16) would never have been made. And the fact that (15) is difficult of integration, and when actually integrated gives an aperiodic discontinuous motion does not seem a very satisfactory justification for making the transformation.

6. **Conclusions.** One of the first inferences of our work is that, whatever we may think of the merits of either of (15) or (16), the Abraham-Planck-Rüdenberg derivation of (16) is without much mathematical or physical validity. The derivation of Lorentz and Ritz[11] is not open to the same objections; for it proceeds directly from the retarded potentials to (16) through a series of approximations. The law of radiation which this equation states is that the rate of radiation is

$$m\frac{dx}{dt}\left(\frac{d^2x}{dt^2}+\kappa^2x\right)=\frac{2}{3}\frac{e^2}{c^3}\frac{dx}{dt}\frac{d^3x}{dt^3},$$

which for oscillatory motions is nearly proportional to the square of the velocity and is physically nearly equivalent to a friction proportional to the velocity. The average rate of radiation is, of course, that which corresponds to that found by Hertz.

Our work throws some light on the method of presenting the differential equation for the oscillator of radiotelegraphy. We may assume that the oscillations are damped simple harmonic; we know that the simplest mathematical representation of such motion is by an equa-

11 Lorentz, Theory of Electrons, loc. cit.; Walther Ritz, Gesammelte Werke, 401 ff.

tion of the second order with constant coefficients, and we may determine the logarithmic decrement of this equation so that the average loss of energy shall be equal to that determined by the theory of Hertz.[12] This is frankly phenomenological and seeks not to go too far below the superficial appearances; but it is probably better than to delve so deeply that we must sacrifice either our skin or our conscience in getting back to the surface.

We have thus far confined our attention to the oscillator, but similar results may be obtained for other systems. For instance, J. J. Thomson[13] has proposed as a model of the atom a center repelling a corpuscle with a force varying as the inverse cube of the distance (and furthermore attracting as the inverse square if the corpuscle lies within a tube issuing from the center). To treat the motion under the repelling force he writes

$$m\frac{d^2x}{dt^2} = \frac{ce}{x^3},$$

integrates the equation and then calculates the radiation by

$$const. \int \left(\frac{d^2x}{dt^2}\right)^2 dt.$$

The motion is therefore obtained by ignoring the reaction of the radiation.

If, however, we write the equation for rate of change of energy at each instant, we have an equation of the type

$$\frac{dr}{dt}\left(\frac{d^2r}{dt^2} - \frac{1}{r^3}\right) = -\kappa\left(\frac{d^2r}{dt^2}\right)^2$$

or

$$f = \frac{-v}{2\kappa} \pm \frac{1}{2\kappa}\sqrt{v^2 + \frac{4\kappa v}{r^3}}.$$

For approach to the center, $v < 0, f > 0$; but the acceleration becomes imaginary when $-vr^3 = 4\kappa$, and the motion has a point d'arrêt. Motion away from the center does not have this peculiarity.

The real interest of our work lies, however, not in the critique of current "demonstrations," nor in a tour de force integration of a

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12 See, for instance, J. Fleming, Principles of Electric Wave Telegraphy, chaps. 3, 5 (1906).

13 J. J. Thomson, On the Structure of the Atom, Phil. Mag. (6) **26**, pp. 792–799, Oct. 1913.

certain differential equation, nor in pointing out that the integration of the equation leads to a physical point d'arrêt, but in the general inferences or alternative inferential possibilities of importance to physical science which the discovery of the point d'arrêt may suggest.

It is clear that we shall not have to abandon the statement of physical problems in terms of differential equations possessing mathematically continuous solutions in order to introduce into physics discontinuities even more startling than those introduced by the adherents of current theories of quanta. The statement of simple cases of rectilinear motion in equations of the second degree and second order has led to cusps in the space-time locus and, by virtue of the irreversibility of time, to physical discontinuities. (Such a conclusion could hardly be drawn from Borel's work, for his equation led to no point d'arrêt.)

If we regard the existence of the point d'arrêt as inconceivable, we must look about for a method of doing away with it. One possibility is to accept literally the method of calculation employed by J. J. Thomson as cited above, namely, to regard the motion of the electric charge as determined by an ordinary equation of mechanics, which omits altogether the reaction of the radiation, and to compute the radiation subsequently. This would mean that the radiant energy did not come out of the field or motion (potential and kinetic energy) of the system but was a phenomenon inherently arising when charges moved.[14] This would be a shock to those who have implicit confidence in the conservation of energy as now generally understood, but would not perhaps be so serious to those who take the position of Ritz[15]: La localisation de l'énergie doit donc être comptée au nombre des conceptions logiquement inutiles (et peut-être parfois nuisibles) de la théorie... Dans le cas le plus général du rayonnement électromagnetique, la conservation de l'énergie n'est plus une loi, mais une convention.

Another method of escape would be to deny the accuracy of the law that the radiation is proportional at each instant to the square of the acceleration. The law is indeed only an approximation to the general form given by Abraham and Heaviside (with the apparent assumption of the quasi-stationary state) and proved by Lewis and me (on the basis of the principle of relativity) to be rigorously true for the point electron moving with any velocity and acceleration,[16]— for with the

14 This may remotely suggest Bateman's double-barreled gun in his Corpuscular Radiation, Phil. Mag. (6) 26, pp. 579–585, Oct. 1913.

15 Gesammelte Werke, pp. 343, 345.

16 Wilson and Lewis, loc. cit., (5).

usual definitions of field and energy the law becomes merely a geometric theorem in four dimensional non-Euclidean geometry. It is reasonable, however, to suppose that the use of the general form of the law could not remove the difficulty; for the point d'arrêt arises under conditions of relatively minimal velocity. Moreover, a qualitative discussion of the equation which arises under the general law indicates[17] that the difficulty is not removed.

It is tolerably clear that we cannot avoid the point d'arrêt and maintain simultaneously the point electron, ordinary mechanics, and ordinary electromagnetic theory. This conclusion may be welcome to the school that follows Bohr in abandoning both mechanics and electromagnetic theory[18] as hitherto understood even when transformed as suggested by the principle of relativity. And we certainly do not wish to imply that this abandon may not work out satisfactorily.

The conservative, however, will naturally try to work out of the present difficulty by abandoning the point electron, especially as electrons are generally supposed to have magnitude. It is a bit hard to see why the assumption of a very small finite size for the electron must fundamentally vitiate the reasoning which leads to the law of the square of the acceleration; but we shall not go into this question, because we are not acquainted with any derivation of this law which does not in some form practically assume the electron to be a point. Moreover, the work of Lorentz and Ritz, starting with a distributed electron and introducing approximations based on the assumption of small velocities and accelerations, establishes a reaction proportional to the rate of change of the acceleration and thus leads on mechanical principles to a law of radiation (at any rate to a loss of mechanical energy) proportional to the (scalar) product of the velocity and rate of change of acceleration. And it is interesting to note that this reaction and rate of change of energy is actually independent of the size or shape assumed for the electron, that is, may be assumed to hold for the point electron.

Howeverso apologetic one may be in regard to (16), that equation seems somewhat less in need of apologies than (15) or its derivation from (15) in integrated form.

Massachusetts Institute of Technology,
Boston, Mass, *May*, 1914.

17 Just what assumptions in non-Newtonian mechanics should be made to treat this problem it is perhaps difficult to state, and consequently the question is here left as a qualitative judgment of the author.

18 N. Bohr, Phil. Mag., (6) 26 (1913) a series of three articles On the Constitution of Atoms and Molecules, pp. 1, 476, 857.